1. Calculus of Variation

To begin with, we define a positive rotationally symmetric mollifier η with compact support by

$$\eta(x) = \begin{cases} c_n e^{-\frac{1}{1-|x|^2}} & |x| < 1 \\ 0 & |x| \geqslant 1, \end{cases}$$

where c_n is a constant satisfying $\int_{\mathbb{R}^n} \eta(x)dx = 1$. Also, for each $\epsilon > 0$ we define

$$\eta_\epsilon(x) = \tfrac{1}{\epsilon^n}\eta(\tfrac{x}{\epsilon}).$$

Notice that $\eta_\epsilon(x) = 0$ for $|x| \geqslant \epsilon$, and $\int_{\mathbb{R}^n} \eta_\epsilon(x) = 1$.

Theorem 1. *Suppose that a smooth function $u : \overline{\Omega} \to \mathbb{R}$ satisfies*

$$\tfrac{1}{2}\int_\Omega |\nabla u|^2 dx \leqslant \tfrac{1}{2}\int_\Omega |\nabla v|^2 dx, \tag{1}$$

for any smooth function $v : \overline{\Omega} \to \mathbb{R}$ with $u = v$ on $\partial\Omega$. Then, u is a harmonic function.

Proof. Given a point $x_0 \in \Omega$, there exists a small ϵ such that $B_\epsilon(x_0) \subset \Omega$. Then, we define a function $I : \mathbb{R} \to \mathbb{R}$ by

$$I(t) = \tfrac{1}{2}\int_\Omega \Big|\nabla\big(u(x) + t\,\eta_\epsilon(x - x_0)\big)\Big|^2 dx.$$

Then, the function $v_t(x) = u(x) + t\eta_\epsilon(x - x_0)$ is smooth and satisfies $u(x) = v_t(x)$ on $\partial\Omega$. Therefore, by the inequality (1) we have $I(0) \leqslant I(t)$. Therefore, $I'(0) = 0$. On the other hand

$$I(t) = \tfrac{1}{2}\int_\Omega |\nabla u|^2 + 2t\langle\nabla u, \eta_\epsilon\rangle + t^2|\nabla\eta_\epsilon|^2 dx.$$

Hence,

$$I'(t) = \int_\Omega \langle\nabla u, \eta_\epsilon\rangle + t|\nabla\eta_\epsilon|^2 dx.$$

Therefore,

$$0 = I'(0) = \int_\Omega \langle\nabla u, \eta_\epsilon\rangle dx = \int_{\partial\Omega} \eta_\epsilon u_\nu dx - \int_\Omega \eta_\epsilon \mathrm{div}(\nabla u) dx$$

Since $|x| > \epsilon$ on $\partial\Omega$, we have $\eta_\epsilon(x - x_0) = 0$ on $\partial\Omega$. Therefore, by recalling $\Delta u = \mathrm{div}(\nabla u)$

$$0 = -\int_\Omega \eta_\epsilon(x - x_0)\Delta u(x) dx$$

Since $\int \eta_\epsilon dx = 1$, we have

$$\Delta u(x_0) = \int_\Omega \eta_\epsilon(x - x_0)\Delta u(x_0)dx - \int_\Omega \eta_\epsilon(x - x_0)\Delta u(x)dx.$$

Thus,

$$|\Delta u(x_0)| \leqslant \int_\Omega \eta_\epsilon(x - x_0)\big|\Delta u(x_0) - \Delta u(x)\big|dx \leqslant \sup_{B_\epsilon(x_0)} \big|\Delta u(x_0) - \Delta u(x)\big|.$$

Since u is smooth, Δu is also smooth. Thus, by passing $\epsilon \to 0$ we have $|\Delta u(x_0)| \leqslant 0$. Hence, $\Delta u(x) = 0$ for all $x \in \Omega$. □

Theorem 2. *Suppose that smooth functions $u : \overline{\Omega} \to \mathbb{R}$ and $f : \overline{\Omega} \to \mathbb{R}$ satisfies*

$$\int_\Omega \tfrac{1}{2}|\nabla u|^2 + fudx \leqslant \int_\Omega \tfrac{1}{2}|\nabla v|^2 + fvdx, \tag{2}$$

for any smooth function $v : \overline{\Omega} \to \mathbb{R}$ with $u = v$ on $\partial\Omega$. Then, $\Delta u = f$ holds in Ω.

Proof. Given a point $x_0 \in \Omega$, there exists a small ϵ such that $B_\epsilon(x_0) \subset \Omega$. Then, we define a function $I : \mathbb{R} \to \mathbb{R}$ by

$$I(t) = \int_\Omega \tfrac{1}{2}\Big|\nabla\big(u(x) + t\,\eta_\epsilon(x - x_0)\big)\Big|^2 + f(x)\big(u(x) + t\,\eta_\epsilon(x - x_0)\big)dx.$$

Then, the function $v_t(x) = u(x) + t\,\eta_\epsilon(x - x_0)$ is smooth and satisfies $u(x) = v_t(x)$ on $\partial\Omega$. Therefore, by the inequality (2) we have $I(0) \leqslant I(t)$. Therefore, $I'(0) = 0$. On the other hand

$$I(t) = \int_\Omega \tfrac{1}{2}|\nabla u|^2 + t\langle \nabla u, \eta_\epsilon\rangle + \tfrac{1}{2}t^2|\nabla \eta_\epsilon|^2 + fu + t\,f\eta_\epsilon dx.$$

Hence,

$$I'(t) = \int_\Omega \langle \nabla u, \eta_\epsilon\rangle + t|\nabla \eta_\epsilon|^2 + f\eta_\epsilon dx.$$

Therefore,

$$0 = I'(0) = \int_\Omega \langle \nabla u, \eta_\epsilon\rangle + f\eta_\epsilon dx = \int_{\partial\Omega} \eta_\epsilon u_\nu dx - \int_\Omega \eta_\epsilon \mathrm{div}(\nabla u)dx + \int_\Omega f\eta_\epsilon dx$$

Since $|x| > \epsilon$ on $\partial\Omega$, we have $\eta_\epsilon(x - x_0) = 0$ on $\partial\Omega$. Therefore, by recalling $\Delta u = \mathrm{div}(\nabla u)$

$$0 = \int_\Omega \eta_\epsilon(x - x_0)\big[- \Delta u(x) + f(x)\big]dx$$

Since $\int \eta_\epsilon dx = 1$, we have

$$\Delta u(x_0) - f(x_0) = \int_\Omega \eta_\epsilon(x - x_0)\big[\Delta u(x_0) - f(x_0)\big]dx - \int_\Omega \eta_\epsilon(x - x_0)\big[\Delta u(x) - f(x)\big]dx.$$

Thus,

$$|\Delta u(x_0) - f(x_0)| \leqslant \int_\Omega \eta_\epsilon(x - x_0)\big|\Delta u(x_0) - f(x_0) - \Delta u(x) + f(x)\big|dx \leqslant \sup_{B_\epsilon(x_0)} \big|\Delta u(x_0) - \Delta u(x)\big| + \big|f(x_0) - f(x)\big|.$$

Since Δu and f are smooth, passing $\epsilon \to 0$ yields $|\Delta u(x_0) - f(x_0)| \leqslant 0$. Hence, $\Delta u = f$ in Ω. □

2. Harnack inequality

We prove the Harnack inequality for more general elliptic PDEs by modifying the proof of Lemma 1.32 in ”Elliptic PDEs” by Han and Lin.

The proof of the following theorem would be too difficult to this intro course.

A key idea to the proof below is to take $\log u$ by using the condition $u > 0$.

Theorem 3. *Let $a_{ij}(x)$ be a smooth symmetric matrix function satisfying*

$$\lambda|\xi|^2 \leqslant a_{ij}(x)\xi_i\xi_j, \tag{3}$$

for $\xi \in \mathbb{R}^n$ where $0 < \lambda$. Suppose that $u(x) : B_1(0) \to \mathbb{R}$ is a positive smooth function to the elliptic equation

$$\mathcal{L}u = a_{ij}(x)\nabla_i\nabla_j u(x) = 0. \tag{4}$$

Then, there exists some constant C depending on $n, \lambda, a_{ij}(x)$ such that

$$\sup_{B_{1/2}(0)} u(x) \leqslant C \inf_{B_{1/2}(0)} u(x). \tag{5}$$

Proof. We define functions

$$w(x) = \log u(x), \qquad Z(x) = |\nabla w(x)|^2, \qquad \eta(x) = 1 - |x|^2$$

We will compute $\mathcal{L}\eta^4 Z = a_{ij}\nabla_i\nabla_j Z\eta^4$ and observe the point where $Z\eta^4$ attains its maximum. Remind that we have $Z\eta^4 = 0$ on $\partial B_1(0)$, and thus $Z\eta^4$ must attain its maximum at an interior point of $B_1(0)$.

To begin with, we compute $\nabla_i w = u^{-1}\nabla_i u$ and thus

$$\nabla_i\nabla_j w = u^{-1}\nabla_i\nabla_j u - u^{-2}\nabla_i u\nabla_j u = u^{-1}\nabla_i\nabla_j u - \nabla_i w\nabla_j w.$$

We multiply by a_{ij} and sum them up. Then, $a_{ij}\nabla_i\nabla_j u = 0$ implies

$$a_{ij}\nabla_i\nabla_j w = u^{-1}a_{ij}\nabla_i\nabla_j u - a_{ij}\nabla_i w\nabla_j w = -a_{ij}\nabla_i w\nabla_j w. \tag{6}$$

Differentiate by ∇_k.

$$\begin{aligned}&\nabla_k a_{ij}\nabla_i\nabla_j w + a_{ij}\nabla_i\nabla_j\nabla_k w = \nabla_k(a_{ij}\nabla_i\nabla_j w) = -\nabla_k(a_{ij}\nabla_i w\nabla_j w)\\ &= -\nabla_k a_{ij}\nabla_i w\nabla_j w - a_{ij}\nabla_k\nabla_i w\nabla_j w - a_{ij}\nabla_k\nabla_j w\nabla_i w = -\nabla_k a_{ij}\nabla_i w\nabla_j w - 2a_{ij}\nabla_k\nabla_i w\nabla_j w.\end{aligned}$$

We remind that the last term $-2a_{ij}\nabla_k\nabla_i w\nabla_j w$ is obtained. Hence,

$$a_{ij}\nabla_i\nabla_j\nabla_k w = -2a_{ij}\nabla_k\nabla_i w\nabla_j w - 2\nabla_k a_{ij}\nabla_i w\nabla_j w. \tag{7}$$

Next, we compute $\mathcal{L}Z$. First of all, we have

$$\nabla_i Z = 2\nabla_i\nabla_k w\nabla_k w. \tag{8}$$

Differentiate it again.

$$\nabla_i\nabla_j Z = 2\nabla_i\nabla_j\nabla_k w\nabla_k w + 2\nabla_i\nabla_k w\nabla_j\nabla_k w.$$

Hence,

$$\mathcal{L}Z = 2a_{ij}\nabla_i\nabla_j\nabla_k w\nabla_k w + 2a_{ij}\nabla_i\nabla_k w\nabla_j\nabla_k w.$$

Combining with (7) and (8) yields

$$2a_{ij}\nabla_i\nabla_j\nabla_k w\nabla_k w = -4a_{ij}\nabla_k w\nabla_k\nabla_i w\nabla_j w - 4\nabla_k a_{ij}\nabla_k w\nabla_i w\nabla_j w = -2a_{ij}\nabla_i Z\nabla_j w - 4\nabla_k a_{ij}\nabla_k w\nabla_i w\nabla_j w.$$

Since a_{ij} is smooth, we have $|\nabla_k a_{ij}| \leqslant C$ for some constant C. Thus,

$$2a_{ij}\nabla_i\nabla_j\nabla_k w\nabla_k w \geqslant -2a_{ij}\nabla_j w\nabla_i Z - C|\nabla^2 w||\nabla w|,$$

for some constant C depending on $|\nabla a|$ and the dimension n. Therefore,

$$\mathcal{L}Z + 2a_{ij}\nabla_j w\nabla_i Z \geqslant 2a_{ij}\nabla_i\nabla_k w\nabla_j\nabla_k w - C|\nabla^2 w||\nabla w|.$$

In addition, (3) implies

$$a_{ij}\nabla_i\nabla_k w\nabla_j\nabla_k w = \sum_k\left[\sum_{i,j} a_{ij}\nabla_i\nabla_k w\nabla_j\nabla_k w\right] \geqslant \sum_k \lambda|\nabla(\nabla_k w)|^2 = \lambda\sum_k\sum_i|\nabla_i(\nabla_k w)|^2 = \lambda|\nabla^2 w|^2.$$

Hence,

$$\mathcal{L}Z + 2a_{ij}\nabla_j w\nabla_i Z \geqslant 2\lambda|\nabla^2 w|^2 - C|\nabla^2 w||\nabla w|.$$

The AM-GM inequality yields

$$\lambda|\nabla^2 w|^2 - C|\nabla^2 w||\nabla w| \geqslant -C|\nabla w|^2 = -CZ.$$

for some constant C. Therefore,

$$\mathcal{L}Z + 2a_{ij}\nabla_j w\nabla_i Z \geqslant \lambda|\nabla^2 w|^2 - CZ.$$

On the other hand, we may denote $\sup_{x\in\Omega}\max_{i,j}|a_{ij}(x)|$ by Λ. Then,

$$|\nabla^2 w|^2 = \sum_{i,j}|\nabla_i\nabla_j w|^2 \geqslant \Lambda^{-2}\sum_{i,j}|a_{ij}\nabla_i\nabla_j w|^2 \geqslant \frac{1}{n^2\Lambda^2}\left|\sum_{i,j}a_{ij}\nabla_i\nabla_j w\right|^2.$$

The last inequality is the Cauchy-Schwarz inequality. Thus, (6) and (3) yield

$$|\nabla^2 w|^2 \geqslant \frac{1}{n^2\Lambda^2}\left|\sum_{i,j}a_{ij}\nabla_i w\nabla_j w\right|^2 \geqslant \frac{1}{n^2\Lambda^2}\left|\lambda^2\sum_i|\nabla_i w|^2\right|^2 = \frac{\lambda^2}{n^2\Lambda^2}Z^2.$$

We denote $\frac{\lambda^3}{n^2\Lambda^2}$ by δ. Then,

$$\mathcal{L}Z + 2a_{ij}\nabla_j w\nabla_i Z \geqslant \delta Z^2 - CZ.$$

Now, we multiply by η^4. By using $|a_{ij}|, \eta, |\nabla\eta|, |\nabla^2\eta| \leqslant C$ we can obtain

$$\begin{aligned}
\mathcal{L}\eta^4 Z &= a_{ij}\nabla_i\left[4\eta^3 Z\nabla_j\eta + \eta^4\nabla_j Z\right]\\
&= a_{ij}\left[12\eta^2 Z\nabla_i\nabla_j\eta + 4\eta^3 Z\nabla_i\nabla_j\eta + 4\eta^3\nabla_i\eta\nabla_j Z + 4\eta^3\nabla_j\eta\nabla_i Z + \eta^4\nabla_i\nabla_j Z\right]\\
&\geqslant -C\eta^2 Z + 8\eta^3 a_{ij}\nabla_i\eta\nabla_j Z + \eta^4\mathcal{L}Z.
\end{aligned}$$

Hence,

$$\mathcal{L}\eta^4 Z \geqslant -C\eta^2 Z + 8\eta^3 a_{ij}\nabla_i\eta\nabla_j Z - 2a_{ij}\eta^4\nabla_j w\nabla_i Z + \delta\eta^4 Z^2 - C\eta^4 Z.$$

Observe

$$8\eta^3 a_{ij}\nabla_i\eta\nabla_j Z = 8\eta^{-1}a_{ij}\nabla_i\eta\nabla_j(\eta^4 Z) - 32Z\eta^2 a_{ij}\nabla_i\eta\nabla_j\eta \geqslant 8\eta^{-1}a_{ij}\nabla_i\eta\nabla_j(\eta^4 Z) - C\eta^2 Z,$$

and

$$-2a_{ij}\eta^4\nabla_j w\nabla_i Z = -2a_{ij}\nabla_j w\nabla_i(\eta^4 Z) + 8a_{ij}\eta^3 Z\nabla_j w\nabla_i\eta \geqslant -2a_{ij}\nabla_j w\nabla_i(\eta^4 Z) - C\eta^3 Z^{\frac{3}{2}}.$$

In the last inequality, we used $Z^{\frac{1}{2}} = |\nabla w|$. Therefore,

$$\mathcal{L}\eta^4 Z \geqslant 8\eta^{-1}a_{ij}\nabla_i\eta\nabla_j(\eta^4 Z) - 2a_{ij}\nabla_i w\nabla_j(\eta^4 Z) + \delta\eta^4 Z^2 - C\eta^3 Z^{\frac{3}{2}} - C\eta^2 Z.$$

For simplicity, we define

$$-b_j(x) = 8\eta^{-1}a_{ij}\nabla_i\eta - 2a_{ij}\nabla_i w.$$

Then, we have

$$\mathcal{L}\eta^4 Z + b_j\nabla_j(\eta^4 Z) \geqslant \delta\eta^4 Z^2 - C\eta^3 Z^{\frac{3}{2}} - C\eta^2 Z.$$

Now, we remind that $\eta^4 Z = 0$ on $\partial B_1(0)$ and $\eta^4 Z > 0$ in $B_1(0)$. Therefore, there exists a point $x_0 \in B_1(0)$ such that $\eta^4 Z(x_0) = \max_{B_1(0)}\eta^4 Z$. Then, at the maximum point, we have $\nabla_j\eta^4 Z(x_0) = 0$. In addition, as Lemma 4 in notes for Feb 28, we can obtain $\mathcal{L}\eta^4 Z(x_0) = a_{ij}\nabla_i\nabla_j\eta^4 Z(x_0) \leqslant 0$. In conclusion, at the point x_0

$$0 \geqslant \delta\eta^4 Z^2 - C\eta^3 Z^{\frac{3}{2}} - C\eta^2 Z.$$

Since $\eta^4 Z(x_0) > 0$, we can divide by $\eta^2 Z$.

$$0 \geqslant \delta\eta^2 Z - C(\eta^2 Z)^{\frac{1}{2}} - C.$$

Therefore,

$$\eta^2 Z(x_0) \leqslant C.$$

Since $\eta \leqslant 1$,

$$\max_{B_1(0)}\eta^4 Z = \eta^4 Z(x_0) \leqslant \eta^2 Z(x_0) \leqslant C.$$

Hence, in the half ball $B_{1/2}(0)$ we have

$$(\tfrac{3}{4})^4 Z(x) \leqslant \eta^4 Z(x) \leqslant C.$$

Thus, $|\nabla w| \leqslant C$ for some constant C in $B_{1/2}(0)$.

We choose two points $x_1, x_2 \in \overline{B_{1/2}}(0)$ such that

$$u(x_1) = \max_{\overline{B_{1/2}}(0)} u(x), \qquad u(x_2) = \min_{\overline{B_{1/2}}(0)} u(x).$$

Then, we define a direction $e = \frac{x_1 - x_2}{|x_1 - x_2|}$, and consider $f(t) = \log u(x_2 + te)$.

$$\log u(x_1) - \log u(x_2) = f(|x_1 - x_2|) - f(0) = \int_0^{|x_2 - x_1|} f'(t)dt = \int_0^{|x_2 - x_1|} \langle \nabla \log u(x_2 + te), e \rangle dt$$

Thus,

$$|\log u(x_1) - \log u(x_2)| \leqslant \int_0^{|x_2 - x_1|} |\nabla \log u(x_2 + te)| dt \leqslant \max_{\overline{B_{1/2}}(0)} |\nabla w| \int_0^{|x_2 - x_1|} dt \leqslant \max_{\overline{B_{1/2}}(0)} |\nabla w| \leqslant C.$$

Since $\log u(x_1) - \log u(x_2) = \log \frac{u(x_1)}{u(x_2)}$, we have the desired result $\frac{u(x_1)}{u(x_2)} \leqslant C$. □